Linear Algebra

UNIT-3

LINEAR TRANSFORMATIONS & ORTHOGONALITY

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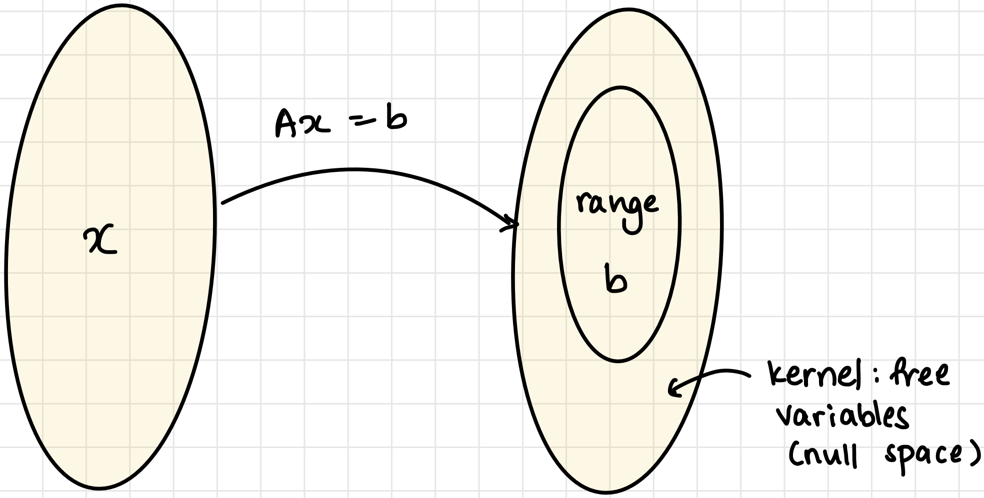
LINEAR TRANSFORMATIONS

- $f : A \to B$ defined by $f(x) = y$
- $A : R^n \to R^m$ (mapping / function)



domain codomain

$C(A) =$ range

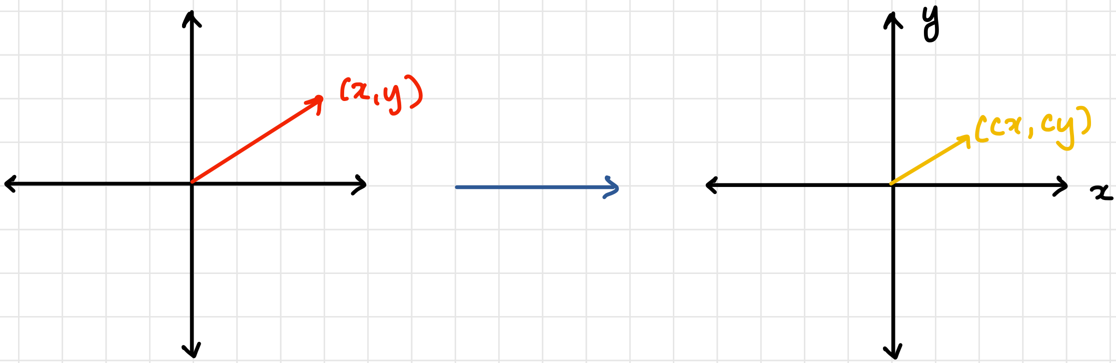
$N(A) =$ free variables / kernel area

Examples

1. $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$ $x = (x, y)$ stretching

$$Ax = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$

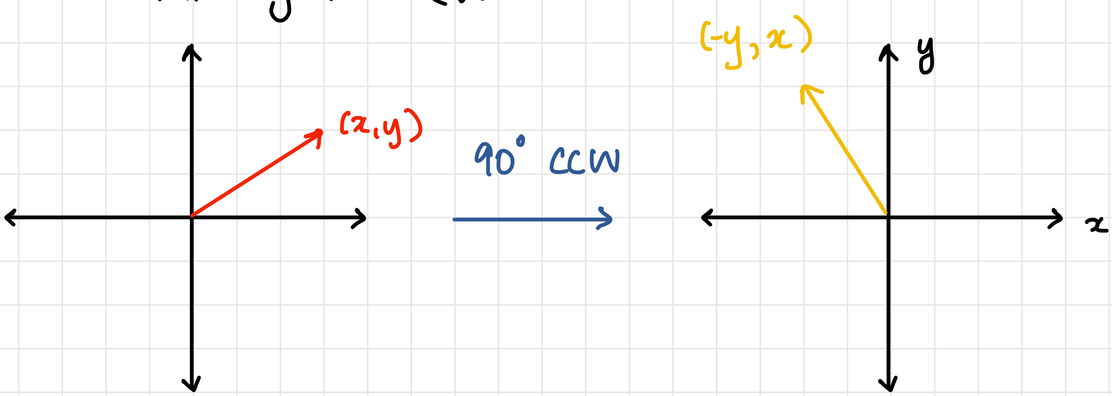
- A multiple of identity matrix $A = cI$ stretches every vector by the scale factor c
- Whole vector space expands or contracts



2. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $x = (x, y)$ Rotation

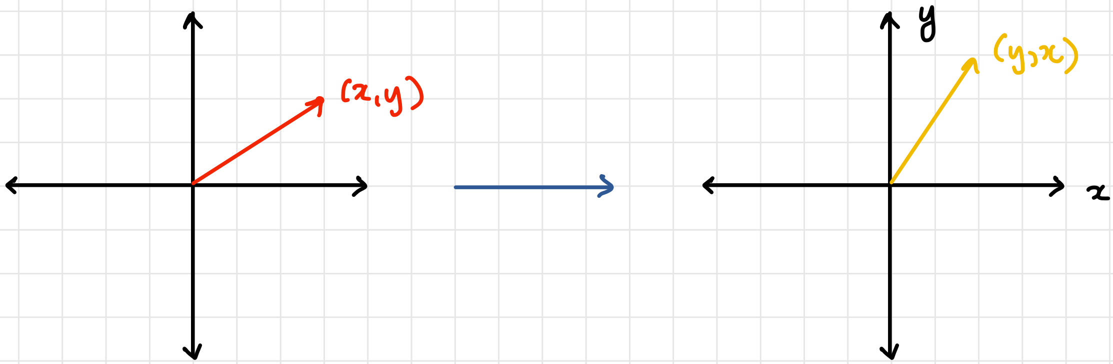
$$Ax = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

- Rotate by 90° CCW



3. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $x = \begin{bmatrix} x \\ y \end{bmatrix}$ Reflection

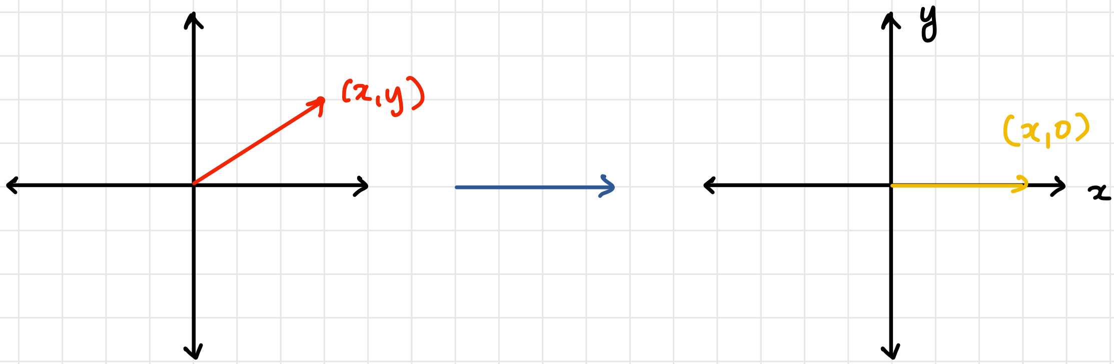
$$Ax = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



- Reflection over $y = x$ (45° mirror)

4. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $x = (x \quad y)$ Projection

$$Ax = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$



- Projection on x-axis

General Matrix to Rotate by Angle θ

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad x = \begin{bmatrix} x \\ y \end{bmatrix}$$

1. $\theta = 0$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. $\theta = \pi/2$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

3. $\theta = \pi$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

RULE of LINEARITY

- A transformation T on R^n is said to be linear if

$$T(cx + dy) = c\,T(x) + d\,T(y)$$

- Preserves origin

Polynomial Space

Space of all polynomials in t of degree n is a vector space denoted by P_n

$$P_n = c_0 + c_1 t + c_2 t^2 + \dots + c_n t^n \mid c_i \in R \}$$

Basis = $(1 \quad t \quad t^2 \quad \ldots \quad t^n)$

Dimension = $n+1$

Examples

1. Differentiation

$A = \frac{d}{dt}$ is linear

- $P_{n+1} \longrightarrow P_n$
- $C(A)$ = all of P_n (n-D area)
- $N(A) = P_0$ (1-D space of all constants)

2. Integration

$A = \int_0^t$ is linear

- $P_n \longrightarrow P_{n+1}$
- $C(A)$ = subspace of P_{n+1}
- $N(A) = Z$

3. Multiplication by Fixed Polynomial

- $A = (3+4t)$
- $A P_n = (3+4t) P_n$

Representation of Polynomial Transformations in Matrix Form

Q1. Construct a matrix associated with differentiation of a polynomial

$$P_3 \longrightarrow P_2$$

Basis $(P_3) = (1 \quad t \quad t^2 \quad t^3)$

Basis $(P_2) = (1 \quad t \quad t^2)$

$$\frac{d}{dt}(1) = 0 = 0(1) + 0(t) + 0(t^2)$$

$$\frac{d}{dt}(t) = 1 = 1(1) + 0(t) + 0(t^2)$$

$$\frac{d}{dt}(t^2) = 2t = 0(1) + 2(t) + 0(t^2)$$

$$\frac{d}{dt}(t^3) = 3t^2 = 0(1) + 0(t) + 3(t^2)$$

$$\overset{A}{\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}}_{3\times4}\ \overset{x}{\begin{bmatrix}\ \\ \ \\ \ \\ \ \end{bmatrix}}_{4\times1} = \overset{b}{\begin{bmatrix}\ \\ \ \\ \ \end{bmatrix}}$$

$$\begin{array}{cccc}1 & t & t^2 & t^3\end{array}$$

$$P(t) = \sqrt{7} - 2\sqrt{3}t + 1.78t^2 + \sqrt{5}t^3$$

$$x = \begin{bmatrix}\sqrt{7}\\-2\sqrt{3}\\1.78\\\sqrt{5}\end{bmatrix}$$

free variable (1)

$$\frac{d}{dt}(P(t)) = \begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}\begin{bmatrix}\sqrt{7}\\-2\sqrt{3}\\1.78\\\sqrt{5}\end{bmatrix} = \begin{bmatrix}-2\sqrt{3}\\3.56\\3\sqrt{5}\end{bmatrix}$$

$$C(A) = \left\{ c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}0\\2\\0\end{bmatrix} + c_3\begin{bmatrix}0\\0\\3\end{bmatrix} \,\middle|\, c_1, c_2, c_3 \in \mathbb{R} \right\}$$

$$\text{Basis}(C(A)) = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\end{bmatrix} \right\}$$

$$\text{Dim}(C(A)) = 3$$

$$N(A) = \left\{ k \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad \text{Dim}(N(A)) = 1 \qquad n - r = 1$$

The multiplication of the matrix A by the polynomial $P(t)$ always yields the derivative of the polynomial.

A is called the differentiation matrix

Q2. Construct a matrix associated with the integration of a polynomial

$$P_2 \longrightarrow P_3$$

$$\text{Basis}(P_2) = \{1, t, t^2\}$$

$$\text{Basis}(P_3) = \{1, t, t^2, t^3\}$$

$$A: \int_0^t \quad dt$$

$$\int_0^t 1\, dt = t = 0(1) + 1(t) + 0(t^2) + 0(t^3)$$

$$\int_0^t t\, dt = \frac{t^2}{2} = 0(1) + 0(t) + \frac{1}{2}(t^2) + 0(t^3)$$

$$\int_0^t t^2\,dt = \frac{t^3}{3} = 0(1) + 0(t) + 0(t^2) + \frac{1}{3}(t^3)$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4\times 3}$$

$$P(t) = 3 + 4t - 6t^2$$

$$\overset{A}{\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}_{4\times 3}} \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ -2 \end{bmatrix}_{4\times 1}$$

$$C(A) = \left\{ c_0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} \;\middle|\; c_1, c_2, c_3 \in R \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} \right\}$$

$$\text{Dim}(C(A)) = 3$$

$$N(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{Dim}(N(A)) = 0 \qquad n - r = 0$$

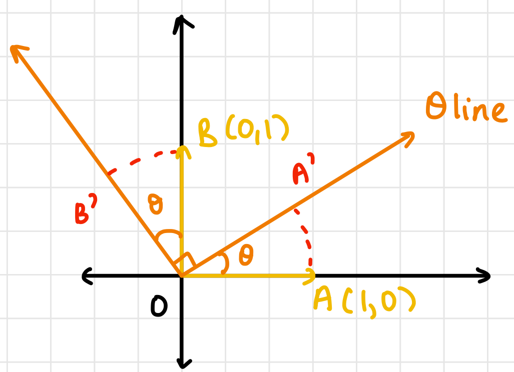
no free variable

Note:
Differentiation is the left inverse of integration

Representation of Transformations in Matrix Form

1. Rotation Q_θ in R^2

- $OA(1,0)$ and $OB(0,1)$ are basis vectors
- Consider rotation Q_θ of the basis vectors by an angle θ in the CCW direction
- Let $A(1,0)$ and $B(0,1)$ be moved to A' and B' respectively
- The new basis vectors are now OA' and OB'
- The coordinates of new bases wrt old bases



$A' = (OA'\cos\theta, OA'\sin\theta)$

$= (\cos\theta, \sin\theta)$

$B' = (OB'\cos(90+\theta), OB'\sin(90+\theta))$

$= (-\sin\theta, \cos\theta)$

$$Q_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

inverse of 2x2 matrix A

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$|Q_\theta| = 1$

- Q_θ is non-singular and hence, invertible

$$Q_\theta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$(Q_\theta^{-1})^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = Q_\theta$$

Rotation by same angle twice

- $Q_\theta \cdot Q_\theta$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = Q_{2\theta}$$

Rotation by 2 angles

- $Q_\theta \cdot Q_\phi$

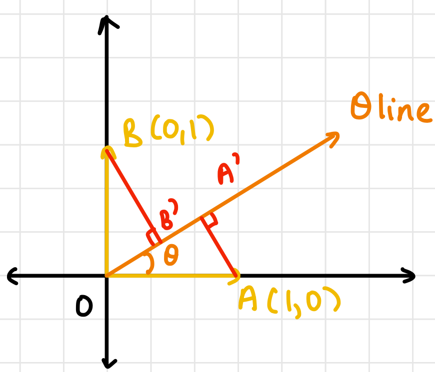
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}$$

2. Projection P in R^2

- Consider projection of R^2 onto θ-line
- Let $A(1,0)$ and $B(0,1)$ get projected onto the theta line as A' and B' respectively



$A' = (OA' \cos\theta, OA' \sin\theta)$ $\qquad$ $B' = (OB' \cos\theta, OB' \sin\theta)$

$= (\cos^2\theta, \cos\theta \sin\theta)$ $\qquad$ $= (\sin\theta \cos\theta, \sin^2\theta)$

$$P = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$|P| = 0$

- P is singular and non-invertible
- There is no way to get original coordinates from the projection (infinitely many)

Projection followed by projection onto same θ line

- $P \cdot P$

Let $c = \cos\theta$
$s = \sin\theta$

$$\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}\begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

$$= \begin{bmatrix} c^4 + c^2s^2 & c^3s + cs^3 \\ c^3s + cs^3 & c^2s^2 + s^4 \end{bmatrix}$$

$$= \begin{bmatrix} c^2(c^2+s^2) & cs(c^2+s^2) \\ cs(c^2+s^2) & s^2(c^2+s^2) \end{bmatrix}$$ $\qquad$ $c^2+s^2=1$

$$= \begin{bmatrix} c^2 & cs \\ cs & s^2 \end{bmatrix}$$

- $P^n = P \quad n = 1, 2, 3, \ldots$

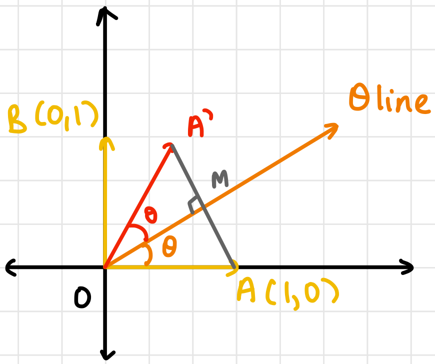
  Projecting any number of times = projecting once

Transpose of P

- $P^T = P$
- $\therefore C(P) = C(P^T)$
- Matrix P is symmetric

3. Reflection H in R^2

- Consider reflection in R^2 on θ-line
- Let A' be the reflection of A on θ line
- Let M be the midpoint of AA'. It is the projection of A on the θ-line



- Consider ΔOAM

$$\vec{OA} + \vec{AM} = \vec{OM} \quad \text{——— (1)}$$

- Consider $\Delta OA'M$

$$\vec{OA'} + \vec{A'M} = \vec{OM} \quad \text{——— (2)}$$

(1) + (2)

$$\vec{OA} + \vec{OA'} + \underbrace{\vec{AM} + \vec{A'M}}_{\vec{AM} + \vec{A'M} = \vec{0}} = 2\vec{OM}$$

(same magnitude, different directions)

$$\vec{OA} + \vec{OA'} = 2\vec{OM}$$

$\vec{OM}$ is projection of OA on the θ-line

$$\vec{x} + H\cdot\vec{x} = 2P\cdot\vec{x}$$

$$\vec{x}\,(I + H\cdot I) = \vec{x}\,(2P\cdot I)$$

Drop $\vec{x}$

$$I + H = 2P$$

$$H = 2P - I$$

$$H = 2\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cos^2\theta - 1 & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & 2\sin^2\theta - 1 \end{bmatrix}$$

$$H = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$|H| = -1$$

- H is non-singular and $\therefore$ invertible

Double Reflection

- $H \cdot H$

$$(2P - I)(2P - I)$$

$$= 4P^2 - 4PI + I^2$$

$$= (4P - 4P + I)$$

$$H^2 = I$$

$$H^{2n} = I$$

Q3. Suppose T is the reflection about 45° line and S is the reflection about Y axis, find in general ST and TS

$$H = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

$$T = \begin{bmatrix} \cos(2\times 45^\circ) & \sin(2\times 45^\circ) \\ \sin(2\times 45^\circ) & -\cos(2\times 45^\circ) \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} \cos(2\times 90^\circ) & \sin(2\times 90^\circ) \\ \sin(2\times 90^\circ) & -\cos(2\times 90^\circ) \end{bmatrix}$$

$$S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$ST = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$TS = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$(ST)^T = TS$$

Q4. Find the matrix of the linear transformation T on R^3 defined by $T(x,y,z) = (2y+z,\ x-4y,\ 3x)$ wrt

i) The standard basis
ii) The basis $\{(1,1,1), (1,1,0), (1,0,0)\}$

i) $T(1,0,0) = (0,1,3)$

$T(0,1,0) = (2,-4,0)$

$T(0,0,1) = (1,0,0)$

$$T = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -4 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

ii) $T(1,1,1) = (3,-3,3)$ with standard bases

$$\begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ using new bases

$c_1 + c_2 + c_3 = 3$

$c_1 + c_2 = -3$

$\boxed{c_1 = 3} \Rightarrow \boxed{c_2 = -6} \Rightarrow \boxed{c_3 = 6} = \begin{bmatrix} 3 \\ -6 \\ 6 \end{bmatrix}$

$T(1,1,0) = (2,-3,3)$ with standard bases

$$\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$c_1 + c_2 + c_3 = 2$

$c_1 + c_2 = -3$

$\boxed{c_1 = 3} \Rightarrow \boxed{c_2 = -6} \Rightarrow \boxed{c_3 = 5} = \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix}$

$T(1,0,0) = (0,1,3)$ wrt standard bases

$$\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ wrt new basis

$c_1 + c_2 + c_3 = 0$

$c_1 + c_2 = 1$

$\boxed{c_1 = 3} \Rightarrow \boxed{c_2 = -2} \Rightarrow \boxed{c_3 = -1} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$

$$T = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$$

NOTE

$$T \neq \begin{bmatrix} 3 & 2 & 0 \\ -3 & -3 & 1 \\ 3 & 3 & 3 \end{bmatrix}$$ as we did not change final coordinates to new basis

Q5. For each of the following LTs T, find the bases and dimension of the range and kernel of T

range → column space

kernel → null space

(i) $T : R^2 \longrightarrow R^3$ given by

$$T(x,y) = (x+y,\ x-y,\ y)$$

(ii) $T : R^2 \longrightarrow R^2$ given by

$$T(x,y) = (y, 0)$$

i) bases of $R^2 = \{ (1,0), (0,1) \}$

bases of $R^3 = \{ (1,0,0), (0,1,0), (0,0,1) \}$

$T(1,0) = (1, 1, 0)$

$T(0,1) = (1, -1, 1)$

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + 1/2 R_2} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} = U$$

$$\rho(T) = 2 = n$$

$$C(T) = \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \mid c_1, c_2 \in R \right\}$$

$$\text{basis}(C(T)) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$\text{Dim}(C(T)) = 2$ D plane in R^3

$$N(T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \because \; n - r = 0$$

$D(N(T)) = 0$

(ii) $T(1,0) = (0,0)$

$T(0,1) = (1,0)$

$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad \rho(T) = 1 \quad n = 2$

$C(T) = \left\{ c \begin{bmatrix} 1 \\ 0 \end{bmatrix}, c \in R \right\}$

basis $(C(T)) = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Dim$(C(T))$ = 1-D line in R^2

$N(T) = Tx = 0$

$y = 0$

$N(A) = \begin{bmatrix} x \\ 0 \end{bmatrix}$ $\quad$ Dim$(N(A)) = 1$

$N(A) = \left\{ k \begin{bmatrix} 1 \\ 0 \end{bmatrix}, k \in R \right\}$

Q6. Construct a matrix that transforms (1,0) to (3,5) and (0,1) to (2,4). Also find the matrix that helps to come back to the original bases (inverse)

$T(1,0) = (3,5)$

$T(0,1) = (2,4)$

$$T = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \Rightarrow T^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 2 & -1 \\ -5/2 & 3/2 \end{bmatrix}$$

Q7. For each of the following LTs find a basis and dimension of the range and kernel of T

$\hookrightarrow C(A)$ $\hookrightarrow N(A)$

(i) $T: R^3 \longrightarrow R^2$ defined by $T(x,y,z) = (x+y, y-z)$

(ii) $T: R^2 \longrightarrow R^2$ defined by $T(x,y) = (x+2y, 2x-y)$

i) $T(1,0,0) = (1,0)$
$T(0,1,0) = (1,1)$
$T(0,0,1) = (0,-1)$

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

free: 1
pivot: 2

$\rho(T) = 2$ $n = 3$

$$C(T) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid c_1, c_2 \in R \right\}$$

$$\text{Dim}(C(T)) = 2$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$N(T) = Tx = 0$$

$$y - z = 0 \qquad x + y = 0$$

$$y = z \qquad x = -z$$

$$N(T) = \begin{bmatrix} -z \\ z \\ z \end{bmatrix} = \left\{ z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \mid z \in R \right\}$$

$$\text{Dim}(N(T)) = 1$$

$$\text{Basis}(N(T)) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$C(T^T) = \left\{ c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \,\middle|\, c_1, c_2 \in R \right\}$$

$$N(T^T) = \quad T^T x = 0$$

$$T^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\rho(T^T) = 2 \qquad n = 2$$

$$\therefore N(T^T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(ii) $T(x,y) = (x+2y, 2x-y)$

$T(1,0) = (1, 2)$
$T(0,1) = (2, -1)$

$$T = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix}$$

$$\rho(T) = 2 = n$$

$$C(T) = \left\{ c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} \mid c_1, c_2 \in R \right\}$$

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$$

$$\text{Dim}(C(T)) = 2$$

$$N(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{Dim}(N(T)) = 0$$

Q8. Find the matrix of LT T on R^3 defined by

$T(x, y, z) = (x+2y+z,\ 2x-y,\ 2y+z)$ wrt

i) Standard basis vectors

ii) Basis: $\{(1,0,1), (0,1,1), (0,0,1)\}$

(i) $T(1,0,0) = (1, 2, 0)$

$T(0,1,0) = (2,-1, 2)$

$T(0,0,1) = (1, 0, 1)$

$$T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

(ii) $T(1, 0, 1) = (2, 2, 1)$ using old basis

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{c_1 = 2}$ $c_1 + c_2 + c_3 = 1$

$\boxed{c_2 = 2}$ $\boxed{c_3 = -3}$

$T(0,1,1) = (3,-1,3)$

$$\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{c_1 = 3}$ $\boxed{c_2 = -1}$ $c_1 + c_2 + c_3 = 3$

$\boxed{c_3 = 1}$

$T(0, 0, 1) = (1, 0, 1)$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\boxed{c_1 = 1}$ $\boxed{c_2 = 0}$ $c_1 + c_2 + c_3 = 1$

$\boxed{c_3 = 0}$

$$T = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

Q9. Let T be LT that sends each matrix x to Ax where $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $V \rightarrow$ set of all 2×2 real matrices.

$x \in V$

Find the matrix that represents T

$T(x) = Ax$

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in R \right\}$$

T: V to V

$$\text{Basis} = \left\{ \underset{v_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{v_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{v_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

$$x = AV = Av_1 + Av_2 + Av_3 + Av_4$$

$$1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Av_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = (1, 0, 1, 0)$$

$$Av_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = (0, 1, 0, 1)$$

$$Av_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = (1, 0, 1, 0)$$

$$Av_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = (0, 1, 0, 1)$$

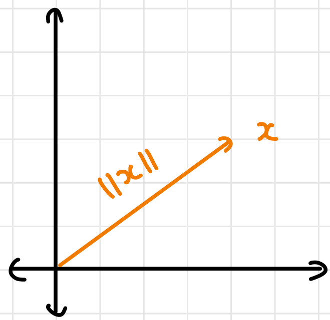
$$T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

ORTHOGONAL VECTORS

(1) Norm
(2) Inner Product
(3) Orthogonal Subspaces

(1) NORM

- Length of a vector
- Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$



- $||x|| \longrightarrow$ norm x

$$||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2} \geq 0$$

$$||x||^2 = x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2 \geq 0$$

$$= x_1 \cdot x_1 + x_2 \cdot x_2 + \dots + x_n \cdot x_n \geq 0$$

$$= [x_1 \ x_2 \ x_3 \cdots x_n] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

$$(||x||)^2 = x^T x$$

Note:

- $||x||$ is the distance of the point from the origin
- $||x|| = 0$ iff $x = \vec{0}$

(2) INNER PRODUCT

- Let $x = (x_1 \quad x_2 \quad \dots \quad x_n)$
- Let $y = (y_1 \quad y_2 \quad \dots \quad y_n)$

$\Big\}$ 2 n-dim vectors

- $\langle x, y \rangle = x^T y = y^T x = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$

Properties of Inner Product

1. if $\langle x, y \rangle > 0$, angle between x & y is acute
2. if $\langle x, y \rangle < 0$, angle between x & y is obtuse
3. if $\langle x, y \rangle = 0$, x and y are orthogonal

(a) $\vec{0}$ is the only vector orthogonal to itself

$\langle 0, 0 \rangle = 0$ or $0^T 0$

(b) $\vec{0}$ is the only vector orthogonal to every other vector

$\langle 0, x \rangle = 0 \ \forall x$ or $0^T x = 0$

(c) $\vec{0}$ is the only vector whose length is $\vec{0}$

(3) ORTHOGONAL SUBSPACES

- Let V be a vector space and S and T be subspaces of V
- We can say that S and T are orthogonal to each other if

$$x^T y = 0 \quad \forall x \in S$$
$$\forall y \in T$$

- In other words, every vector s in S is orthogonal to every vector t in T

Examples

(a) $V = \{0\}$, $S = \{0\}$, $T = \{0\}$

$$\langle S, T \rangle = 0$$

(b) $V = R^1$, $S = \{0\}$, $T =$ subspace of R^1

$$\langle S, T \rangle = 0$$

Note:

1. If $\mathrm{Dim}(V) = n$, then $\mathrm{Dim}(S) + \mathrm{Dim}(T) \leq n$
2. If S and T are orthogonal, then $S \cap T = \vec{0} = Z$

THEOREM 1

If nonzero vectors $v_1, v_2, v_3 \dots v_n$ are mutually orthogonal, then these vectors are linearly independent

Mutually orthogonal : $v_i^T v_j = 0 \ \forall \ i \neq j$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n = 0 \longrightarrow (1)$$

To prove : $c_i = 0 \ \forall \ i$

Multiply (1) by v_1^T

$$c_1 v_1^T v_1 + \cancel{c_2 v_1^T v_2}^{\,0} + \cancel{c_3 v_1^T v_3}^{\,0} + \dots + \cancel{c_n v_1^T v_n}^{\,0} = 0$$

$$c_1 \ ||v_1|| = 0$$

$$||v_1|| \neq 0 \quad \text{(nonzero vector)}$$

$$\therefore c_1 = 0$$

Similarly, for all v_i^T,

$$c_i = 0$$

Generally,

$$\therefore \ c_1 v_1 + c_2 v_2 + c_3 v_3 + \dots + c_n v_n = 0$$

$$\text{iff} \quad c_i = 0 \ \forall \ i$$

THEOREM 2 : FUNDAMENTAL THEOREM OF ORTHOGONALITY

Let A be a matrix of order $m \times n$, then

1) $C(A^T)$ and $N(A)$ are orthogonal subspaces in R^n

2) $C(A)$ and $N(A^T)$ are orthogonal subspace in R^m

Proof

1) Suppose x is a vector in the null space. Then $Ax = 0$ and system of m equations can be written as:

$$Ax = \begin{bmatrix} \cdots & \text{row } 1 & \cdots \\ \cdots & \text{row } 2 & \cdots \\ & \vdots & \\ \cdots & \text{row } m & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Row 1 is orthogonal to x (inner product = 0)
- Every row is orthogonal to x
- x orthogonal to every combination of rows
- Each x in the null space is perpendicular to each vector in the row space

$$N(A) \perp C(A^T)$$

2) Suppose y is a vector in the left null space. Then $A^T y = 0$ system of m equations can be written as:

$$A^T y = \begin{bmatrix} \cdots \text{column } 1 \cdots \\ \vdots \\ \cdots \text{column } n \cdots \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = 0$$

- Every column is orthogonal to y
- y is orthogonal to every combination of columns

$$N(A^T) \perp C(A)$$

Orthogonal Complement

Let V be a vector space. The set of all vectors orthogonal to every vector in V is called orthogonal complement of V

$$V^{\perp} \rightarrow V \text{ perp}$$

$\therefore$ the largest set of vectors becomes the orthogonal complement

eg: XOY is the complement of z

THEOREM 3: FUNDAMENTAL THEOREM OF LINEAR ALGEBRA, PT 2

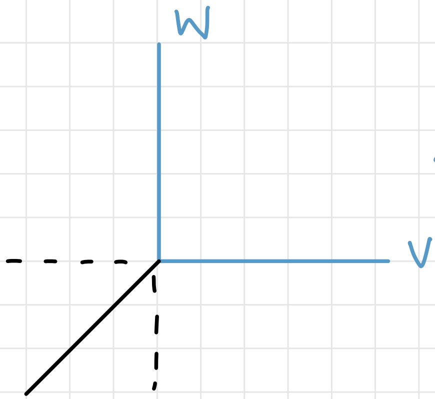
Let A be a matrix of order $m \times n$

1) $C(A^T)$ = complement of $N(A)$ in R^n

2) $C(A)$ = complement of $N(A^T)$ in R^m

They are orthogonal and complementary subspaces

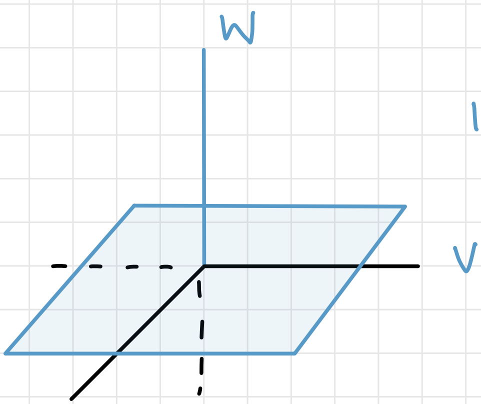
ORTHOGONAL BUT NOT ORTHOGONAL COMPLEMENTS



$W \perp V$ in $\mathbb{R}^3$

two orthogonal axes

ORTHOGONAL COMPLEMENTS



line $W \perp$ to plane V in $\mathbb{R}^3$

$$V^{\perp} = W \quad \text{and} \quad W^{\perp} = V$$

Properties

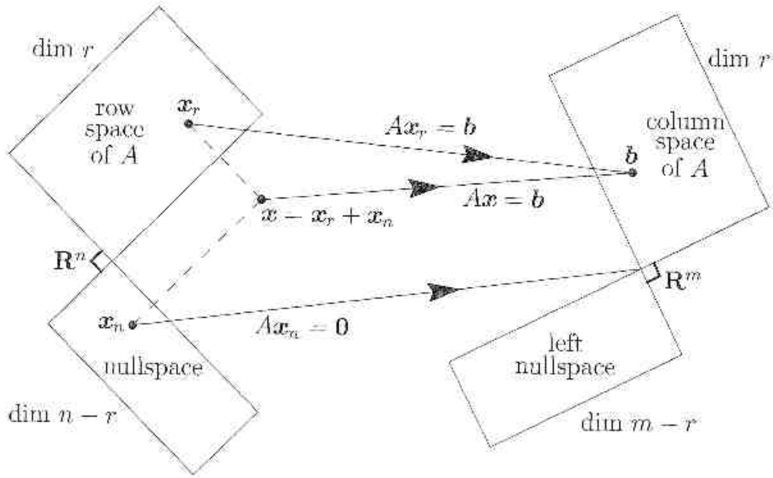
(i) $(V^{\perp})^{\perp} = V$

ii) If $V^{\perp} = W$, $W^{\perp} = V$

iii) If V and W are orthogonal complements in R^n, then

$$\dim(V) + \dim(W) = n = \dim R^n$$

The following figure summarises the effect of matrix multiplication



- Everything from row space goes to column space
- Everything from null space goes to origin
- Splitting R^n into orthogonal parts V and W will split every vector into $x = v + w$
  - vector v is projection of x onto subspace V
  - orthogonal component w is the projection of x onto W

- The true effect of matrix multiplication is that
  - every Ax is in column space
  - null space goes to 0
  - row space component goes to column space
  - nothing is carried to left null space

- Every Ax transforms row space to column space

Q10. Find the lengths and inner product of $x = (1,4,0,2)$ and $y = (2,-2,1,3)$

$$\|x\| = \sqrt{1+16+4} = \sqrt{21} \qquad \|y\| = \sqrt{4+4+1+9} = \sqrt{18}$$

$$\langle x, y \rangle = x^T y = 2 - 8 + 0 + 6 = 0$$

Q11. Which pairs of vectors are orthogonal?

$v_1 = (1,2,-2,1)$ $v_2 = (4,0,4,0)$ $v_3 = (1,-1,-1,-1)$ $v_4 = (1,1,1,1)$

$\langle v_1 v_2 \rangle = 4 + 0 - 8 + 0 = -4$

$\langle v_1 v_3 \rangle = 1 - 2 + 2 - 1 = 0$ ✓

$\langle v_1 v_4 \rangle = 1 + 2 - 2 + 1 = 2$

$\langle v_2 v_3 \rangle = 4 + 0 - 4 + 0 = 0$ ✓

$\langle v_2 v_4 \rangle = 4 + 0 + 4 + 0 = 0$

$\langle v_3 v_4 \rangle = 1 - 1 - 1 - 1 = -2$

Q12. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}$ be a matrix.

(a) Find a vector x which is orthogonal to row space of A

(b) Find a vector y which is orthogonal to column space of A

(c) Find a vector z which is orthogonal to null space of A

(a) Null space : $Ax = 0$ (Null space $\perp$ row space)

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$z = 0$ $\qquad x + 2y = 0$

let $y = k$ $\qquad x = -2k$ $\qquad z = 0$

$$N(A) = \left\{ \begin{bmatrix} -2k \\ k \\ 0 \end{bmatrix} \middle| k \in R \right\}$$

$$N(A) = \left\{ k \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \middle| k \in R \right\}$$

$\dim(A) = 1$ $\qquad$ basis $= \{ (-2 \;\; 1 \;\; 0) \}$

(b) Left null space (Column space ⊥ Left null space)

$$A:b = \left[\begin{array}{ccc:c} 1 & 2 & 1 & b_1 \\ 2 & 4 & 3 & b_2 \\ 3 & 6 & 4 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 2 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 1 & b_3 - 3b_1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc:c} 1 & 2 & 1 & b_1 \\ 0 & 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right]$$

$$N(A^T) = \left\{ k \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, k \in R \right\}$$

or

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$y + z = 0$

Let $z = k$

$y = -k$

$x + 2y + 3z = 0$

$x - 2k + 3k = 0$

$x + k = 0$

$x = -k$

$$N(A^T) = \left\{ k \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, k \in R \right\}$$

(c) Vectors with pivot variables ⇒ row space

$$z = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{or} \quad z = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

ORTHONORMALITY

Set of nonzero vectors are said to be orthonormal if

(i) $v_i^T v_j = 0 \, , \; i \neq j$

(ii) $\|v_i\| = 1 \quad \forall \, i$

$$v_i^T v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

In other words, $v_i^T v_j = 0$

Eg:

(i) $(1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2})$

Note:

The coordinate vectors i.e. the vectors that lie on the x-axis, are orthonormal in R^n.

In particular, if $e_1 = (1,0), e_2 = (0,1)$ are orthonormal in R^2

If the vectors are rotated through θ, then the the new vectors $(\cos\theta, \sin\theta)$ and $(-\sin\theta, \cos\theta)$ are also orthonormal

Q13. Find all vectors in R^3 that are orthogonal to a $(1,1,1)$ and b $(1,-1,0)$. Construct an orthonormal basis from these vectors

- 2 vectors form plane
- Find line perpendicular to plane

Let u be a vector in R^3 $\perp$ to $(1,1,1)$ and $(1,-1,0)$.

$$u = (x \quad y \quad z)$$

$$u^T a = 0 = u^T b$$

$$x + y + z = 0 \quad \text{and} \quad x - y = 0$$

$$[A:b] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right]$$

$$-2y - z = 0$$

Let $y = k$ $\qquad\qquad x + k - 2k = 0$

$z = -2k$ $\qquad\qquad x = k$

$$\therefore u = \begin{bmatrix} k \\ k \\ -2k \end{bmatrix} = \left\{ k \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \middle| k \in R \right\}$$

$v_1 = a = (1, 1, 1)$ $v_2 = b = (1, -1, 0)$

$v_3 = (1, 1, -2)$

(orthonormal to v_1 & v_2)

From independent orthonormal vectors, produce basis by dividing each vector by its norm to make unit vectors

Normalising v_1, v_2, v_3 vectors will get orthonormal bases

$(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$

$(1/\sqrt{2}, 1/\sqrt{2}, 0)$

$(1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6})$

Q14. Let P be the plane in R^4 $x - 2y + 3z - t = 0$

(i) Find a vector $\perp$ to P
(ii) What matrix has the plane P as its null space?
(iii) What is the basis for P?

(i) $P = x - 2y + 3z - t = 0$ is a 3D plane in R^4

$$\underset{P^T}{\begin{bmatrix} 1 & -2 & 3 & -1 \end{bmatrix}} \underset{v}{\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}} = 0$$

$(1, -2, 3, -1)$ is dir ratio $\perp$ to plane

$$\therefore\ v = \left\{ k \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix} \middle|\ k \in R \right\} \text{ is } \perp \text{ to } P$$

(ii) Let the matrix A have null space = P

Let $u \in P$ such that $u = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}$

u is left null space

$$u^T P = \begin{bmatrix} x & y & z & t \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = 0$$

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \end{bmatrix}$$

$x - 2y + 3z - t = 0$ is the solution to $Ax = 0$

(iii) Basis for P : basis of null space

null space: solutions to P

$$\begin{bmatrix} 1 & -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x - 2y + 3z - t = 0$$

$$x = 2y - 3z + t$$

$$N(P) = \begin{bmatrix} 2y-3z+t \\ y \\ z \\ t \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

basis = $\{ (2,1,0,0), (-3,0,1,0), (1,0,0,1) \}$

Q15. Suppose S is spanned by $(1,2,2,3)$ and $(1,3,3,2)$. Find the basis for $S^{\perp}$

Let v be a vector in $S^{\perp}$. Let $v = (x,y,z,t)$

$v^T(1,2,2,3) = 0$ and $v^T(1,3,3,2) = 0$

or

Let S = row space of matrix. $S^{\perp}$ = null space of matrix

$$A:b = \left[\begin{array}{cccc:c} 1 & 2 & 2 & 3 & 0 \\ 1 & 3 & 3 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{cccc:c} 1 & 2 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\Big\downarrow R_1 \to R_1 - 2R_2$$

$$R = \left[\begin{array}{cccc:c} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

Let $t = k_1$, $z = k_2$

$$x + 5k_1 = 0$$

$$x = -5k_1$$

$$y + z - t = 0$$

$$y + k_2 - k_1 = 0$$

$$y = k_1 - k_2$$

$$N(V) = \begin{bmatrix} -5k_1 \\ k_1 - k_2 \\ k_2 \\ k_1 \end{bmatrix} = \left\{ k_1 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; k_1, k_2 \in \mathbb{R} \right\}$$

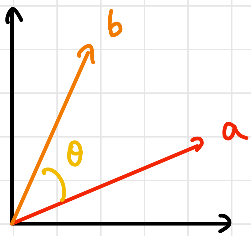
$\therefore$ Basis for $S^\perp = \left\{ \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}$

COSINES & PROJECTIONS

If $a = (a_1, a_2)$, $b = (b_1, b_2)$ angled θ apart, then

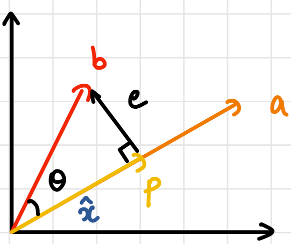
$$\cos\theta = \frac{a_1 b_1 + a_2 b_2}{\|a\| \|b\|} = \frac{a^T b}{\|a\| \|b\|}$$

Applies to R^n



Projections onto Line

Projection of $\vec{b}$ onto line a



$e = b - p$

$\vec{p}$ is a multiple of a, point closest to $\vec{b}$ on a

$$p = \hat{x} a$$

multiple (scalar)

$$a \perp e \quad \text{or} \quad a^T(b - \hat{x}a) = 0$$

$$\hat{x} a^T a = a^T b$$

$$\boxed{\hat{x} = \frac{a^T b}{a^T a}}$$

$p = a\hat{x}$

$$\boxed{p = a\,\frac{a^T b}{a^T a}}$$

$p = Pb$ P: projection matrix

$$\boxed{P = \frac{a\,a^T}{a^T a}}$$ → $n \times n$ matrix, symmetric $P^T = P$

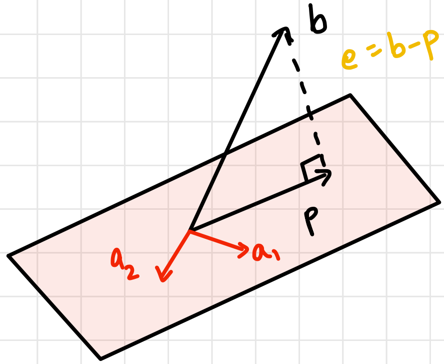
$C(P)$ = line through a

PROJECTION MATRIX IS OF RANK 1

$r(P) = 1$ (column vector x row vector)

- Note: $P^n = P$ (property)

Project vector onto space



plane of a_1, a_2

= column space of

$$\begin{bmatrix} \vdots & \vdots \\ a_1 & a_2 \\ \vdots & \vdots \end{bmatrix}$$

$e \perp$ plane spanned by a_1 & a_2

p is some multiple of a_1 and a_2

$$p = \hat{x}_1 a_1 + \hat{x}_2 a_2$$

$$\boxed{p = A\hat{x}}$$

$$p = \begin{bmatrix} \vdots & \vdots \\ a_1 & a_2 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$e = b - A\hat{x} \perp$ plane $\Rightarrow \perp$ to a_1 & $\perp$ to a_2

$$a_1^T(b - A\hat{x}) = 0 \quad \text{and} \quad a_2^T(b - A\hat{x}) = 0$$

Writing equations into matrix form

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}_{2\times 1} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T \underbrace{(b - A\hat{x})}_{e} = 0 \longrightarrow (1)$$

e is in $N(A^T)$

$e \perp C(A) \longrightarrow$ plane

Rewrite eq (1)

$$\boxed{A^T A\hat{x} = A^T b} \longrightarrow (2)$$

Solve for $\hat{x}$

$$\hat{x} = (A^T A)^{-1} A^T b$$

Projection p

$$p = A\hat{x}$$

if b is in $C(A)$, $p = b$ and if b is in $N(A^T)$ then $p = 0$

$$\boxed{p = A(A^T A)^{-1} A^T b}$$

projection vector closest to b

$$\boxed{P = A(A^T A)^{-1} A^T}$$

projection matrix

In 1-D

$$p = \frac{a a^T}{a^T a} b$$

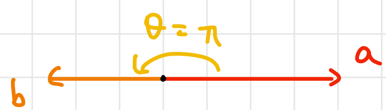
note: A is not square; cannot do $(A^T A)^{-1} = A^{-1}(A^T)^{-1}$ as A^{-1} does not exist

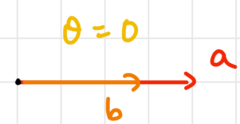
SCHWARZ INEQUALITY

All vectors a and b in $\mathbb{R}^n$

$$|a^T b| \le \|a\| \|b\| \quad \text{or} \quad |\cos\theta| \le 1$$

If $\theta = 0$ or $\theta = \pi$, equality holds (dependent vectors) and b = projection on a, $e = 0$





Note

1. P is symmetric

2. $P^n = P$ for $n = 1, 2, 3 \ldots$

3. $r(P) = 1$

4. Trace of $P = 1$

5. If a is n-dimensional vector of order n, P is square matrix of order n

6. If a is a unit vector, $P = a a^T$ $(a^T a = 1)$

Q16. What multiple of $a(1,1,1)$ is closest to the point $b(2,4,4)$? Find the point which is closest to a on the line through b.

$$P_a = \hat{x} a \qquad \text{where} \qquad \hat{x} = \frac{a^T b}{a^T a} = \frac{a^T b}{\|a\|^2}$$

$$\hat{x} = \frac{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{10}{3} \text{ multiple}$$

$$P_a = \hat{x} a = 10/3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P_b = \hat{x}\, b \qquad \hat{x} = \frac{b^T a}{b^T b} = \frac{\begin{bmatrix} 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}}$$

$$P_b = \frac{10}{36}\begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 20/36 \\ 40/36 \\ 40/36 \end{bmatrix} = \begin{bmatrix} 5/9 \\ 10/9 \\ 10/9 \end{bmatrix}$$

Q17. Find the projection of b onto a

(i) $a = (1, 0)$, $b = (c, s)$

(ii) $a = (1, -1)$, $b = (1, 1)$

(i) $a = (1, 0)$, $b = (\cos\theta, \sin\theta)$

$$P_a = \hat{x}\, a = \frac{a^T b}{a^T a}\, a = \frac{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}}{\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \cos\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$P_a = \begin{bmatrix} \cos\theta \\ 0 \end{bmatrix}$$

(ii) $a = (1, -1)$, $b = (1, 1)$

$$P_a = a\hat{x} = \frac{a^T b}{a^T a}\, a = \frac{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Q.18. If P is a plane of vectors in R^4

$$P \equiv u+v+w+t=0$$

Find P and $P^\perp$ (null space of P)

(i) P

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ t \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$u+v+w+t=0$$

$$u=-v-w-t$$

$$P = N(A) = \left\{ \begin{bmatrix} -v-w-t \\ v \\ w \\ t \end{bmatrix} \right\}$$

$$= \left\{ v \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; v, w, t \in R \right\}$$

(ii) $P^\perp$. Row space is $(\text{Null space})^\perp$

$$P^\perp = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \;\middle|\; k \in R \right\}$$

Q19. Let S be a 2D subspace in R^3 spanned by $a = (1,2,1)$, $b = (1,-1,1)$. Write the vector $v = (-2,2,2)$ as the sum of a vector in S and a vector orthogonal to S.

vector in $S \in$ column space of S
vector in $S^\perp \in$ left null space of S

find row space

$$A^T = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 0 \end{bmatrix}$$

$$\Big\downarrow R_2 \to -\tfrac{1}{3} R_2$$

$$R^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xleftarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \text{row space} = \left\{ k_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \;\middle|\; k_1, k_2 \in R \right\}$$

$S^\perp$ = left null space (solution to $A^T x = 0 = R^T x$)

$x + z = 0$ $\qquad$ $y = 0$
$x = -z$

left null space $S^\perp = \left\{ k \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; k \in R \right\}$ = line $\perp$ to plane

$$\therefore \; V = \overbrace{C_1 V_1 + C_2 V_2}^{\text{vector in } S} + \overbrace{C_3 V_3}^{\text{vector in } S^\perp}$$

where V_1, V_2 are bases of $C(A^T)$ and V_3 is basis of $N(A)$

$$v = c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\end{bmatrix} + c_3\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-2\\2\\2\end{bmatrix}$$

$$\begin{bmatrix}c_1 - c_3\\c_2\\c_1 + c_3\end{bmatrix} = \begin{bmatrix}-2\\2\\2\end{bmatrix}$$

OR

Projection of v onto line (left null space)

Let P lie on $S^\perp$

$$P = \frac{a^T v}{a^T a} \cdot a = \frac{\begin{bmatrix}-1 & 0 & 1\end{bmatrix}\begin{bmatrix}-2\\2\\2\end{bmatrix}}{\begin{bmatrix}-1 & 0 & 1\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix}}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \frac{4}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$P = \begin{bmatrix}-2\\0\\2\end{bmatrix}$$

$\therefore$ orthogonal component in $S = v - P$

$$v - P = \begin{bmatrix}-2\\2\\2\end{bmatrix} - \begin{bmatrix}-2\\0\\2\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix}$$

$\therefore\ v = (0,2,0) + (-2,0,2)$

Q20. Project $b = (1, 0, 0)$ onto the lines

(i) $a_1 = (-1, 2, 2)$
(ii) $a_2 = (2, 2, -1)$
(iii) $a_3 = (2, -1, 2)$

Add the three points of projections and explain what the sum is and why it is.

(i) $$P_1 = \frac{a^Tb}{a^Ta} \cdot a = \frac{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}} \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = -1/9 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

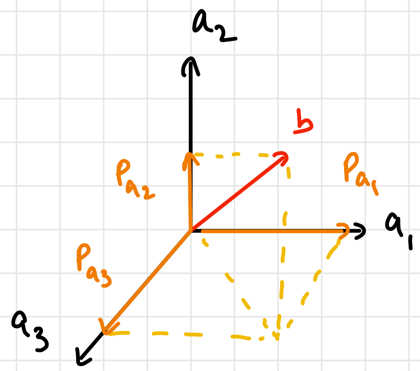
(ii) $$P_2 = \frac{a^Tb}{a^Ta} \cdot a = \frac{\begin{bmatrix} 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 2 & 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 2/9 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

(iii) $$P_3 = \frac{a^Tb}{a^Ta} \cdot a = \frac{\begin{bmatrix} 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}} \cdot \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 2/9 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$P_1 + P_2 + P_3 = \frac{1}{9}\begin{bmatrix} 1 + 4 + 4 \\ -2 + 4 - 2 \\ -2 - 2 + 4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = b$$

Since a_1, a_2 and a_3 are mutually orthogonal,
$P_1 + P_2 + P_3 = (1,0,0) = b$

We bring the original vector back



Q21. V is a subspace of R^5 spanned by $a = (1,2,3,-1,2)$ and $b = (2,4,7,2,-1)$. Find a basis of the orthogonal comp. $V^\perp$.

Let $s \in V^\perp = (v, w, x, y, z)$. $As^T = 0$

$$\begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 2 & 4 & 7 & 2 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & -1 & 2 \\ 0 & 0 & 1 & 4 & -5 \end{bmatrix}$$

$\Big\downarrow R_1 \to R_1 - 3R_2$

$$R = \begin{bmatrix} 1 & 2 & 0 & -13 & 17 \\ 0 & 0 & 1 & 4 & -5 \end{bmatrix}$$

Let $y = k_1$, $z = k_2$ Let $v = k_3$

$x + 4y - 5z = 0$ $u + 2v - 13k_1 + 17k_2 = 0$

$x = -4k_1 + 5k_2$ $u = -2k_3 + 13k_1 - 17k_2$

$$\therefore V^\perp = \left\{ \begin{bmatrix} -2k_3 + 13k_1 - 17k_2 \\ k_3 \\ -4k_1 + 5k_2 \\ k_1 \\ k_2 \end{bmatrix} \right\}$$

$$V^\perp = \left\{ k_1 \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} + k_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

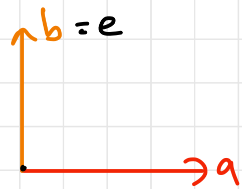
$$\text{Basis}(V^\perp) = \left\{ \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$\text{Dim}(V^\perp) = 3$

Q22. Project $b = (1, 2, 2)$ onto the line through $a = (2, -2, 1)$. Check if e is perpendicular to a.

$$P = \frac{a^T b}{a^T a} \cdot a = \frac{[2\ \ -2\ \ 1]\begin{bmatrix}1\\2\\2\end{bmatrix}}{9}\begin{bmatrix}2\\-2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore\ b \perp a$



$$e = b - P = b$$

$$\langle e, a \rangle = 2 - 4 + 2 = 0$$

Q23. Project $b = (1, 2, 2)$ onto the line through $a = (1, 1, 1)$. Check if $e \perp a$

$$P = \frac{a^T b}{a^T a} \cdot a = \frac{[1\ \ 1\ \ 1]\begin{bmatrix}1\\2\\2\end{bmatrix}}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{5}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$e = b - P = \begin{bmatrix}1\\2\\2\end{bmatrix} - \begin{bmatrix}5/3\\5/3\\5/3\end{bmatrix} = \begin{bmatrix}-2/3\\1/3\\1/3\end{bmatrix}$$

$$\langle e, a \rangle = \frac{-2}{3} + \frac{1}{3} + \frac{1}{3} = 0 \quad \therefore\ e \perp a$$

PROJECTIONS & LEAST SQUARES

Failure of Gaussian elimination with multiple equations and one variable (b not in C(A))

$$\begin{aligned} a_1 x &= b_1 \\ a_2 x &= b_2 \\ a_3 x &= b_3 \end{aligned} \qquad \text{or} \qquad Ax = b$$

Solvable if $a_1 : a_2 : a_3 = b_1 : b_2 : b_3$

If system is inconsistent, choose value of x that minimises average error E in the m equations.

$$\text{Sum of squares} = E^2 = \sum_{i=1}^{m} (a_i x - b_i)^2$$

If there is exact solution, $E = 0$. If not, $\frac{dE^2}{dx} = 0$

Solving for x

$$\frac{dE^2}{dx} = \sum_{i=1}^{m} 2(a_i x - b_i) a_i = 2 \sum_{i=1}^{m} a_i^2 x - 2 \sum_{i=1}^{m} a_i b_i = 0$$

$$\sum_{i=1}^{m} a_i^2 x = \sum_{i=1}^{m} a_i b_i$$

$$a^T a (x) = a^T b$$

$$\boxed{\hat{x} = \frac{a^T b}{a^T a}}$$

$\hat{x} = a^{-1} p$

Least Squares WITH Multiple Variables

- Consider an inconsistent system of linear equations

$$A_{m \times n} X_{n \times 1} = b_{m \times 1}$$

- We look for best possible approximate solution.
- The vector b lies outside C(A) and we need to project it onto C(A) to get the point p in C(A) closest to b
- The system is reduced to $A\hat{x} = p$

From pages 49,50

$$\boxed{A^T A \hat{x} = A^T b} \rightarrow \text{normal equation}$$

solve for $\hat{x}$ (estimate)

- The equation $A^T A \hat{x} = A^T b$ is called the normal equation

Q24. Find $\|E\|^2 = \|Ax-b\|^2$ and set to zero its derivatives wrt the unknowns u and v. Compare the resulting equation with the normal equation

$$A^T \cdot A\hat{x} = A^T \cdot b$$

(i) Find the solution $\hat{x}$ and the projection $p = A\hat{x}$

(ii) Why is $p = b$?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad x = \begin{bmatrix} u \\ v \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

Using least squares method

$$\|E\|^2 = \|Ax-b\|^2$$

$$Ax-b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$Ax-b = \begin{bmatrix} u-1 \\ v-3 \\ u+v-4 \end{bmatrix}$$

$$\|E\|^2 = \left\| \begin{matrix} u-1 \\ v-3 \\ u+v-4 \end{matrix} \right\|^2 = (u-1)^2 + (v-3)^2 + (u+v-4)^2$$

Derivative wrt u

$$\frac{\partial \|E\|^2}{\partial u} = 2(u-1) + 2(u+v-4) = 0$$

$$u+1 + u+v-4 = 0$$

$$2u + v - 3 = 0$$

$$2u+v = 3 \longrightarrow (1)$$

Derivative wrt v

$$\frac{\partial \|E\|^2}{\partial v} = 2(v-3) + 2(u+v-4) = 0$$

$$v-3 + u+v-4 = 0$$

$$2v + u - 7 = 0$$

$$2v+u = 7 \longrightarrow (2)$$

Using geometry

$$A^T A \hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$2u+v=5$ and $u+2v=7$

$\therefore$ the equations are the same

(i) Solution $\hat{x}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$A = \left[\begin{array}{cc:c} 2 & 1 & 5 \\ 1 & 2 & 7 \end{array}\right] \xrightarrow{R_2 \to R_2 - 1/2 R_1} \left[\begin{array}{cc:c} 2 & 1 & 5 \\ 0 & 3/2 & 9/2 \end{array}\right]$$

$$\frac{\cancel{3}}{\cancel{2}}v = \frac{\cancel{9}^{3}}{\cancel{2}} \Rightarrow \boxed{v=3}$$

$$2u+3=5 \Rightarrow \boxed{u=1}$$

Solution: $\hat{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

(ii) Projection $p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$p = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = b$$

$p = b \Rightarrow b$ is in column space of A

Q25. Let $A = [3 \ 1 \ -1]$. Let $V = N(A)$. Find

(i) A basis for V, basis for $V^\perp$
(ii) Projection matrix P_1 onto $V^\perp$
(iii) Projection matrix P_2 onto V

$V = N(A)$ = solution to $Ax = 0$ where $x = (x, y, z)$

$3x + y - z = 0$ $\quad [3 \ 1 \ -1] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [0]$

$x = \frac{z-y}{3}$

$$N(A) = \left\{ \begin{bmatrix} (z-y)/3 \\ y \\ z \end{bmatrix} \right\}$$

$$V = N(A) = \left\{ y \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \,\middle|\, y, z \in R \right\}$$

(i) Basis for $V = \left\{ \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Basis for $V^\perp$ = basis for row space

$$= \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(ii) P_1 onto $V^\perp$

$$P = A(A^T A)^{-1} A^T$$

$$P_1 = V^\perp ((V^\perp)^T V^\perp)^{-1} (V^\perp)^T$$

$$P_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \left(\begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \left(\begin{bmatrix} 11 \end{bmatrix}^{-1} \right) \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \left(\frac{1}{11} \right) \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{1\times 1} \underbrace{\begin{bmatrix} 3 & 1 & -1 \end{bmatrix}}_{1\times 3}$$

$$= \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \left(\frac{1}{11}\right) \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$$

$$P_1 = \frac{1}{11} \begin{bmatrix} 9 & 3 & -3 \\ 1 & 3 & -1 \\ -3 & -1 & 1 \end{bmatrix}$$

(iii) P_2 onto V

$$V = \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P_2 = V (V^T V)^{-1} V^T$$

$$= \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 10/9 & -1/9 \\ -1/9 & 10/9 \end{bmatrix}^{-1} \begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\frac{9}{11}\right) \begin{bmatrix} 10/9 & 1/9 \\ 1/9 & 10/9 \end{bmatrix}_{2\times 2} \begin{bmatrix} -1/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix}_{2\times 3}$$

$$= \frac{9}{11} \begin{bmatrix} -1/3 & 1/3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -11/27 & 10/9 & 1/9 \\ -11/27 & 1/9 & 10/9 \end{bmatrix}$$

$$= \frac{9}{11} \begin{bmatrix} 0 & -1/3 & 1/3 \\ -11/27 & 10/9 & 1/9 \\ -11/27 & 1/9 & 10/9 \end{bmatrix}$$

Q26. Find projection of b onto the C(A)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

Split b into $p+q$ such that p is in C(A) and q is $\perp$ to that space. Which of the four subspaces contains q?

Column space

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 + 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 6 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$V = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$$C(A) = \left\{ x \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\}$$

Projection p

$$(A^T A)\hat{x} = A^T b$$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -8 \\ -8 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 27 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 6 & -8 & -11 \\ -8 & 18 & 27 \end{array}\right] \xrightarrow{R_2 \to R_2 + 8/6\, R_1} \left[\begin{array}{cc:c} 6 & -8 & -11 \\ 0 & 22/3 & 37/3 \end{array}\right]$$

$$22y = 37 \qquad\qquad 6x - \frac{148}{11} = -11$$

$$y = \frac{37}{22} \qquad\qquad x = \frac{9}{22}$$

$$\hat{x} = \begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix}$$

$$p = A\hat{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 9/22 \\ 37/22 \end{bmatrix}$$

$$p = \begin{bmatrix} 9/22 + 37/22 \\ 9/22 + -37/22 \\ -9/11 + 74/11 \end{bmatrix}$$

$$p = \begin{bmatrix} 23/11 \\ -14/11 \\ 65/11 \end{bmatrix}$$

$$q = b - p = \begin{bmatrix} 1 - 23/11 \\ 2 + 14/11 \\ 7 - 65/11 \end{bmatrix}$$

$$q = \begin{bmatrix} -12/11 \\ 36/11 \\ 12/11 \end{bmatrix}$$

q is in null space of A^T